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# PARAMETRIC EXCITATION OF FLEXURAL-GRAVITY EDGE WAVES IN THE FLUID BENEATH AN ELASTIC ICE SHEET WITH A CRACK

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**Abstract** – The properties of elastic-gravity oscillations of deep water beneath a thin elastic plate with a crack are investigated in the paper. The dependence of the reflection and transition coefficients of the waves through the crack on wave frequency and incident angle are found. The shape of the fluid surface deformed by edge waves, propagating along the crack and decreasing exponentially away from the crack, is investigated in the vicinity of the crack. The asymptotic equations describing the parametric excitation of counterpropagating edge waves by flexural-gravity waves which hit the crack at normal incidence are derived. 

## Introduction

This work is devoted to the investigation of the properties of flexural-gravity waves on the surface of deep water beneath a thin elastic plate with a crack. It is assumed that this model reflects some properties of flexural-gravity oscillations of floating ice cover. Ice cracks are typical structural irregularities of sea ice cover (Zubov, 1943). They are formed permanently in the ice under the influence of internal stresses, induced by external mechanical (wind, water waves, pressure from surrounding ice cover) and thermodynamic factors. The investigation, carried out in the paper, gives answers on the questions about the degree of influence of one rectilinear crack in the ice on plane flexural-gravity wave and about the existence of new modes of flexural-gravity waves in this system.

As a rule, the appearance of new wave modes in inhomogeneous and unbounded media is related to the formation of open waveguides. The waveguides can be formed in the regions where the group velocity of the waves, propagating in the waveguide direction, has a local minimum. The ocean shelf zone and underwater ridge are typical examples of open waveguides for surface gravity waves (Stokes, 1846; Ursell, 1951). Waveguides modes of surface waves are named edge waves. The energy of the edge waves is localized in the vicinity of the waveguide and the amplitude of the edge waves is decreasing exponentially away from the waveguide. Akylas (1983), Miles (1991), Pierce and Knobloch (1994) have investigated the interactions of counterpropagating edge waves in the shelf zone, the wave radiation in the open ocean induced by nonlinear edge wave interactions, and the parametric excitation of counterpropagating edge waves under the influence of gravity waves coming from the open ocean at normal incidence.

All the effects mentioned above are examples of parametric wave interactions in Faraday systems. The first experimental investigations of parametric excitation of capillary-gravity waves by external vibrating sources were carried out by Faraday (1831). The incident waves are analogues of the vibrating source in the problems of parametric excitation of edge waves in open waveguides. The investigation of these effects for the flexural-gravity waves has not been carried out before.

Some difficulties in the investigation of nonlinear interactions of edge waves are related with finding short asymptotic equations, which describe the processes. Using standard techniques of multiscale expansions, which have been used for finding well known equations of KdV or NSE types, is not possible in the case under consideration. It is related with the existence of an additional space scale, which characterizes the damping of edge waves by moving off the waveguide. These difficulties can be overcome by using averaging techniques in the space direction normal to the waveguide. In this work the other approach, based on the Wiener-Hopf methodology, is developed.

Let us mention some work on the linear theory of flexural-gravity waves, propagating in the fluid beneath an inhomogeneous ice cover. The main interest in this field is the calculation of the propagation of surface gravity waves, coming from the open ocean, through the ice edge beneath an ice cover. Waves which penetrate beneath an ice cover can produce the fracture of the ice cover and moreover influence the structure of the marginal zone of drifting sea ice. In the simplest model, the edge of the ice is a horizontal straight line separating the open water from the ice covered water. It is assumed that monochromatic gravity waves come from the open water to the ice edge, interact with it, then partially penetrate beneath the ice cover and partially reflect from the edge. The natural approach for the calculation of this process is the Wiener-Hopf technique (Noble, 1958), which gives the solution of the problem analytically. Solutions of this problem were found by Heins (1948) (the problem about wave diffraction on a dock), Weitz and Keller (1950) (the ice cover is a continuum composed of noninteracting mass points), Evans and Davies (1968) (the ice cover is modeled by a thin elastic plate).

In spite of the Wiener-Hopf methodology the numerical investigation of these analytical solutions is quite complicated. Therefore considerable progress in the investigation of this problem was achieved in another way, relating to the numerical solution of an integral equation for the velocity potential of the fluid. This approach was developed in the work of Fox and Squire (1990, 1994). Later this methodology was used for the investigation of plane gravity wave diffraction on floating elastic bands of finite width (Meylan and Squire, 1994).

The use of the Wiener-Hopf methodology turned out to be most effective for the solution of problems dealing with the propagation of hydroacoustic (Kouzov, 1963) and flexural-gravity (Marchenko, 1993) waves in the fluid beneath a thin elastic plate with rectilinear crack. In this case the factorization in the Wiener-Hopf method is trivial. The solution has a simple analytic form and depends on several free constants. This ambiguity is removed by the definition of the amplitude of the waves, transporting the energy from infinity to the crack, and contact-boundary conditions at the crack edges. It is interesting that solutions can be constructed in this case, when flexural-gravity waves, transporting the energy from infinity to the crack and from the crack into infinity, are absent (Marchenko, 1997a). The condition for the existence of these solutions is the dispersion equation for edge waves, propagating along the crack.

Marchenko and Semenov (1994) proved the existence of edge waves, propagating along the crack in a thin elastic plate, floating on the surface of shallow water. The natural surface oscillations of the water in an ice channel have been investigated in the paper of Marchenko (1997b) also in the approximation of shallow water. The ice channel is modeled by an infinite cut with parallel edges in the floating thin elastic plate. It was shown that the dispersion curve of the first natural mode of the channel is transformed into the dispersion curve of edge waves, propagating along the crack, when the channel width tends to zero. The existence of symmetric edge waves, propagating along a rectilinear crack in an elastic sheet, floating on the surface of deep water, has been proved in the paper of Marchenko (1997a).

In the present paper the properties of known analytical solutions (Marchenko, 1997a), describing the flexural-gravity oscillations of deep water beneath a thin elastic plate with a crack, are investigated numerically. The theory of weakly nonlinear wave interactions is developed on the basis of these analytical solutions. The paper is organized as follows. In § 1 the closed system of equations and boundary conditions, describing the potential motion of the fluid in the system, is formulated. The question about possible forms of contact-boundary conditions at the crack edges is discussed. In § 2 the structure of the solutions of the linearized problem is investigated in the plane of the parameters: wave frequency and wave number in the crack direction. The dependence of the modulus of the transmission coefficient of the wave through the crack on the frequency and the incident angle of the wave is constructed. The shape of the fluid surface, deformed by edge waves in the vicinity of the crack, is investigated numerically for two wave frequencies. In § 3 the technique of asymptotic expansions is used to find simpler equations, describing multimode interactions in the system under consideration is used. The equations, describing the parametric excitation of counterpropagating edge waves by the normal incident flexural-gravity waves, are derived. In the conclusion a possible application of the results obtained in this paper for the description of the wave dynamics of a sea ice cover is discussed.

## 1. Basic equations

In dimensionless variables the full system of equations, describing the potential motions of deep water beneath an elastic plate, consists of the Laplace equation for the velocity potential of the fluid

$$(\Delta + \frac{\partial^2}{\partial z^2})\varphi = 0, \ z < \varepsilon\eta, \tag{1.1}$$

the boundary conditions on the lower surface of the plate

$$\frac{\partial \eta}{\partial t} + \varepsilon \nabla\eta\nabla\varphi = \frac{\partial\varphi}{\partial z}, \ \frac{\partial\varphi}{\partial t} + \frac{\varepsilon}{2}((\nabla\varphi)^2 + (\frac{\partial\varphi}{\partial z})^2) + \eta + p - p_a = 0, \ z = \varepsilon\eta \tag{1.2}$$

and the condition of vanishing of fluid motion at large depth: $\varphi \to 0 \ \ z \to -\infty$.

Here the following symbols are considered: $\varphi$ is the velocity potential of the fluid, $z = \varepsilon\eta(t, x, y)$ is the equation of the fluid surface, $p$ is the pressure in the fluid beneath the elastic plate, $p_a$ is the external atmosphere pressure, $x$, $y$ are the horizontal coordinates, $z$, $t$ are the vertical coordinate and the time, $\Delta = \nabla^2$ is two dimensional Laplace operator, $\nabla = (\partial/\partial x, \partial/\partial y)$. The small dimensionless parameter $\varepsilon$ is equal to the ratio of typical wave amplitude $a$ to typical wave length $l$.

Assumed that the change of elastic energy of the plate $\mathcal{F}$ is equal to the work of the pressure

$$\delta\mathcal{F} = \int (p - p_a)\delta\eta dxdy. \tag{1.3}$$

This equation is not explicit, since the change of kinetic energy of the plate is not taken into account. The motivation of this assertion is related to the assumption, that the kinetic energy change of the plate is smaller than the change of fluid kinetic energy in the system under consideration. This assumption is satisfied if the typical horizontal scale of the motion and the fluid depth are much larger than the thickness of the plate.

The elastic energy of the plate is equal to

$$\mathcal{F} = \mathcal{F}_h + \mathcal{F}_n, \tag{1.4}$$

where

$$2\mathcal{F}_h = \int D(\Delta\eta)^2 dxdy, \ \mathcal{F}_n = \int D(1-\nu)\left[\left(\frac{\partial^2\eta}{\partial x\partial y}\right) - \frac{\partial^2\eta}{\partial x^2}\frac{\partial^2\eta}{\partial y^2}\right] dxdy.$$

The dimensionless rigidity of the elastic plate is defined by the formula

$$D = \frac{\mathcal{D}}{l^4}, \ \mathcal{D} = \frac{Eh^3}{12(1-\nu^2)\rho_w g},$$

where $E$ and $\nu$ are the Young's modulus and the Poisson's ratio of the plate, $h$ is the thickness of the plate, $\rho_w$ is fluid density and $g$ is acceleration due to the gravity. Furthermore it is assumed that $l = \mathcal{D}^{1/4}$ and $D = 1$.

Assume that there is a straight linear crack in the elastic plate, and that the crack line coincides with the horizontal line $x = 0$. Let us calculate the variation $\delta\mathcal{F}$, taking into account that the variation $\delta\eta$ is equal to zero at infinity. We have

$$\delta\mathcal{F} = \int \delta\eta\Delta^2\eta dxdy + A_f^+ + A_f^- + A_m^+ + A_m^-, \tag{1.5}$$

where the letters $A_f^\pm$ and $A_m^\pm$ denote the work of transverse shears and bending moments at the crack edges

$$A_f^\pm = \int\limits_{-\infty}^{\infty} F^\pm \delta\eta dy,\ A_m^\pm = \int\limits_{-\infty}^{\infty} M^\pm \frac{\partial\delta\eta}{\partial x} dy.$$

The transverse shears and the bending moments are defined by the following formulas

$$F^\pm = -\lim_{x\to\pm 0} \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} + \nu'\frac{\partial^2}{\partial y^2}\right)\eta,\ M^\pm = -\lim_{x\to\pm 0}\left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\right)\eta \quad (\nu' = 2-\nu).$$

From (1.3) and (1.5) we have

$$p - p_a = \Delta^2\eta. \tag{1.6}$$

If the crack edges are free and external forces and moments do not act on them, then the following contact-boundary conditions hold at the crack edges:

$$F^\pm = M^\pm = 0. \tag{1.7}$$

In the system under consideration the energy dissipation is absent. It is assumed that the source of wave perturbations of the fluid is at infinity and has a constant intensity. Let us define the density of total energy of the fluid and the plate by the following formula

$$\mathcal{E} = \mathcal{P} + \mathcal{T} + \mathcal{F}, \tag{1.8}$$

where the densities of potential $\mathcal{P}$ and kinetic $\mathcal{T}$ energy are equal to

$$\mathcal{P} = \frac{1}{2}\eta^2,\ \mathcal{T} = \frac{1}{2}\int\limits_{-\infty}^{\varepsilon\eta}\left[(\nabla\varphi)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right] dz.$$

The law of total energy balance has the form

$$\frac{\partial\mathcal{E}}{\partial t} = \nabla\cdot\Pi, \tag{1.9}$$

where $\Pi$ is the vector of energy flux

$$\Pi = \int\limits_{-\infty}^{\varepsilon\eta} \nabla\varphi\frac{\partial\varphi}{\partial t}dz + D(\Delta\eta\nabla\frac{\partial\eta}{\partial t} - \frac{\partial\eta}{\partial t}\nabla\Delta\eta) + D(1-\nu)\mathbf{A}, \tag{1.10}$$

$$\mathbf{A} = \left(\frac{\partial^2\eta}{\partial x\partial y}\frac{\partial^2\eta}{\partial t\partial y} - \frac{\partial^2\eta}{\partial y^2}\frac{\partial^2\eta}{\partial t\partial x}, \frac{\partial^2\eta}{\partial x\partial y}\frac{\partial^2\eta}{\partial t\partial x} - \frac{\partial^2\eta}{\partial x^2}\frac{\partial^2\eta}{\partial t\partial y}\right).$$

The vector $\mathbf{A}$ defines the flux of tension energy of the plate, induced by its flexural deformations. One can see that $\mathbf{A}$ is different from zero if the bending deformations are two dimensional. This cannot be realized without the tension of the neutral surface of the plate.

The influence of the plate is the appearance of an additional linear term $\Delta^2\eta$ in the dynamical boundary condition (1.2) and a typical length scale $l$, which defines the typical scale of flexural deformations for which the influence of gravity force as well as elastic forces, acting on the fluid by the plate, are of the same order. The contact-boundary conditions depend on the Poisson's ratio $\nu$ explicitly only if the deformations are two dimensional. This fact follows from the Kirchhoff-Love hypotheses and Hook's law, which have been used to derive the equations of the classic theory of thin elastic plates (Timoshenko and Woinowsky-Kriger, 1959).

The Kirchhoff-Love hypotheses define the structure of flexural deformations inside the plate. In some practical applications the data about the structure of these deformations can be absent. In these cases using these hypotheses is not fully justified. On the other hand the hypothesis that the density of the elastic energy $\mathcal{F}$ of the plate depends on the curvature $\kappa$ of plate surface is more natural. From this, it follows that $\mathcal{F} = D\kappa$ for small flexions of the plate, where the proportionality factor $D$ is the rigidity of the plate and $\kappa \approx \Delta^2\eta$. In this approach the divergent term $\mathcal{F}_n$ is absent in the expression for the total energy density. The associated contact-boundary conditions at the crack edges have the form (1.7), where it is supposed that $\nu = 1$.

Both approaches are equivalent when the flexion of the plate does not depend on $y$. Furthermore the problem dealing with oblique reflexion and transmission of flexural-gravity waves through the crack in an elastic plate will be considered. It will be shown that the properties of the solution of this problem essentially depend on the value of the coefficient $\nu$ in contact-boundary conditions.

## 2. The solution of the linear problem of flexural-gravity oscillations of deep water beneath an elastic plate with a crack

Consider the flexural-gravity oscillations of small amplitude in deep water beneath an elastic plate with a straight linear crack. Assume that the dependence of the velocity potential and the perturbation of fluid surface on $t$ and $y$ is given by the following formulas

$$\varphi = \phi(x,z)e^{i\theta},\ \eta = \zeta(x)e^{i\theta},\ \theta = \omega t - k_y y. \tag{2.1}$$

The equations for finding the functions $\phi$ and $\zeta$ follow from the equations (1.1) and (1.2) by setting $\varepsilon = 0$ and have the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} - k_y^2\right)\phi = 0,\ z < 0, \tag{2.2}$$
$$-\omega^2\phi + (1 + \left(\frac{\partial^2}{\partial x^2} - k_y^2\right)^2\frac{\partial\phi}{\partial z}) = 0,\ i\omega\zeta = \frac{\partial\phi}{\partial z},\ z = 0.$$

Solutions of the equations (2.2), which are bounded and continuous in the region $z < 0$, are written in the region I: $\omega > \omega_*(k_y)$ (see fig.1) in the following form

$$\phi = \frac{i\omega}{\lambda_0}(a^+e^{ik_0x} + a^-e^{-ik_0x})e^{i\lambda_0 z} + \frac{i\omega}{2\pi}\int\limits_{-\infty}^{\infty}\frac{P_3(k)}{\kappa(\omega,\lambda)}e^{ikx+\lambda z}dk, \tag{2.3}$$
$$\zeta = a^+e^{ik_0x} + a^-e^{-ik_0x} + \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\lambda P_3(k)}{\kappa(\omega,\lambda)}e^{ikx}dk$$

and in the region II (see fig.1) in the form

$$\phi = \frac{i\omega}{2\pi}\int\limits_{-\infty}^{\infty}\frac{P_3(k)}{\kappa(\omega,\lambda)}e^{ikx+\lambda z}dk,\ \zeta = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\lambda P_3(k)}{\kappa(\omega,\lambda)}e^{ikx}dk. \tag{2.4}$$

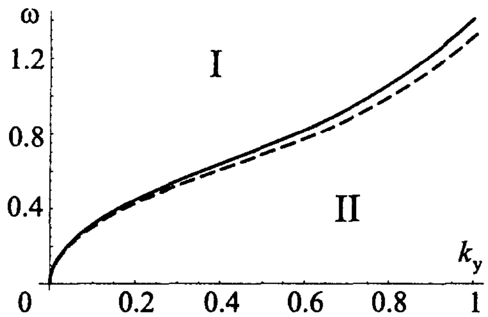


FIGURE 1. The regions I and II of qualitatively different behavior of the solutions, describing the flexural-gravity oscillations of deep water beneath an elastic plate with a crack. The continuous curve is described by the equation $\omega = \omega_*(k_y)$. The dotted curve is related to edge waves.

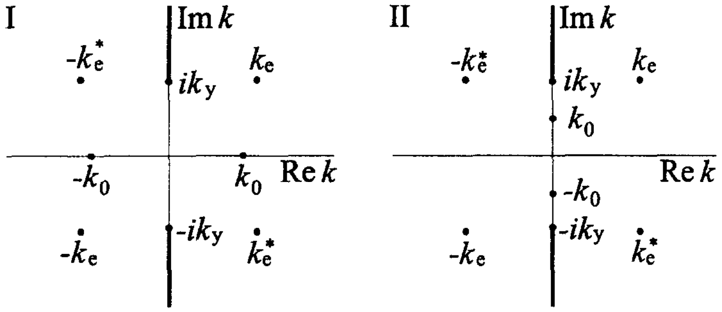


FIGURE 2. The location of the roots of the dispersion relation in the plane of the complex variable $k$, when the frequency $\omega$ is in the regions I and II shown in fig.1. The bold lines denote the cuts of the two-sheeted Riemann surface.

Here the following notation has been used:

$$\omega_*(k_y) = \sqrt{k_y(1+k_y^4)},\ \kappa(\omega,\lambda) = \omega^2 - \lambda(1+\lambda^4),$$

$$\lambda(k) = \sqrt{k^2+k_y^2},\ \lambda_0 = \lambda(k_0),\ P_3(k) = \sum_{n=0}^{3} a_n k^n,\ a_n = \text{const.}$$

In the region I the dispersion equation $\kappa(\omega,\lambda) = 0$ has two real roots $k = \pm k_0$ for any given value of $k_y$. These roots become imaginary when the frequency $\omega$ moves from region I to region II. The existence of the real roots of the dispersion equation means the existence of plane monochromatic waves with frequency $\omega$ and real wave vectors $\mathbf{k}^+ = (k_0, k_y)$ and $\mathbf{k}^- = (-k_0, k_y)$, transporting the energy from infinity to the crack and from the crack to infinity. The dispersion equation in the regions I and II has also four complex roots $k = \pm k_e$ and $k = \pm k_e^*$ for any given value $k_y$. The location of the roots of the dispersion equation in the complex plane $k$ is shown in fig. 2.

The contour of integration in formulas (2.3) and (2.4) coincides with the real axis $k$. The integrals in formulas (2.3) must be interpreted in the sense of principal value. The integrals are calculated by the residues of the integrand expressions in the complex plane $k$. It is necessary to take into account the fact that the function $\lambda(k)$ is defined on a two-sheeted Riemann surface. The sheet, where $\Re\lambda > 0$, is used in the solutions (2.3) and (2.4). The cuts of the sheet coincide with the semiaxis $|\Im k| > k_y$ and are marked in fig.2 by bold lines.

The solutions in the analytic form (2.3) and (2.4) are found by using the Wiener-Hopf methodology (Marchenko, 1993). One can see that the function $\phi(x,z)$ is continuous in the region of fluid motion. The function $\zeta(x)$ can have discontinuities at the point $x=0$. The possibility of the existence of discontinuities is due to the degree of the polynomial $P_3(k)$ and the asymptotics of the integrand expressions as $|k|\to\infty$.

From the formulas (2.3) and (2.4) it follows that the solutions in the regions I and II depend respectively on six and four free constants $a^{\pm}$ and $a_n$. These constants must be chosen so that according to equations (1.7) the contact-boundary conditions

$$\lim_{x\to\pm 0}\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}-\nu' k_y^2\right)\zeta=\lim_{x\to\pm 0}\left(\frac{\partial^2}{\partial x^2}-\nu k_y^2\right)\zeta=0 \tag{2.5}$$

are satisfied.

The formulas (2.3) describe the diffraction of plane flexural-gravity waves on the straight linear irregularity in the elastic plate. In the case where contact-boundary conditions have the form (2.5), the irregularity is the crack. The solution is the superposition of incident and reflected plane waves and diffracted perturbation, which is localized in the vicinity of the irregularity. The amplitudes $A^+$ and $A^-$ of the waves, transporting the energy from $+\infty$ and $-\infty$ respectively, are defined by the formulas

$$A^+=a^++\frac{1}{2}F(k_0),\ A^-=a^--\frac{1}{2}F(-k_0) \tag{2.6}$$

where $F(k)=i\lambda P_3(k)(\partial\kappa/\partial k)^{-1}$.

The amplitudes $B^+$ and $B^-$ of the waves, transporting the energy from the irregularity into $+\infty$ and $-\infty$, are equal to

$$B^+=a^++\frac{1}{2}F(-k_0),\ B^-=a^--\frac{1}{2}F(k_0).$$

The asymptotic behavior of the expressions (2.3) is given by the formulas

$$\phi=\frac{i\omega}{\lambda_0}(A^+e^{ik_0x}+B^+e^{-ik_0x}),\ \zeta=A^+e^{ik_0x}+B^+e^{-ik_0x},\ x\to\infty, \tag{2.7}$$
$$\phi=\frac{i\omega}{\lambda_0}(A^-e^{-ik_0x}+B^-e^{ik_0x}),\ \zeta=A^-e^{-ik_0x}+B^-e^{ik_0x},\ x\to-\infty.$$

One can see that the solution (2.3) at $|x|\to\infty$ is the superposition of real monochromatic waves, propagating in the fluid beneath a homogeneous elastic sheet without irregularities. These waves are described by the formulas (2.1), where

$$\phi=\frac{i\omega}{\lambda(k)}Ae^{ikx+\lambda(k)z},\ \zeta=Ae^{ikx} \tag{2.8}$$

and $k=\pm k_0$. The formulas (2.1) and (2.8) describe complex waves, if $k$ is one of the complex roots of the dispersion equation.

Define the average energy flux of the wave with wave number $k$, related to the $x$ direction, by the formula

$$\langle\Pi_x\rangle=\frac{k_y}{2\pi}\int_0^{2\pi/k_y}\Pi_x dy \tag{2.9}$$

where the vector of energy flux $\Pi$ is defined in (1.10).

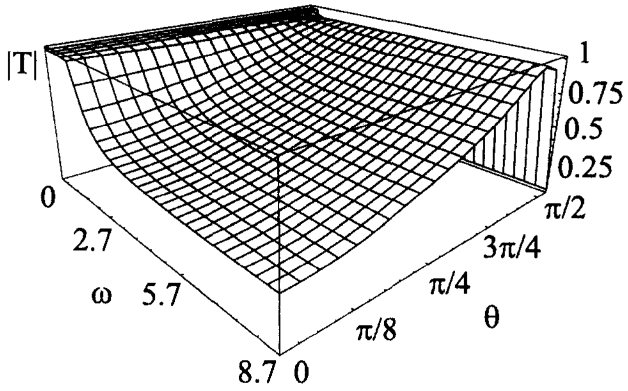


FIGURE 3. The dependence of the modulus of the transmission coefficient on the wave frequency $\omega$ and the angle $\theta$ of wave incidence on the crack. The coefficient $\nu$ is equal to 0.34.

Let us substitute the real part of the solution, defined by the formulas (2.1) and (2.8), in the formulas (1.10) and then in the definition (2.9). With accuracy up to square terms in the amplitude $A$, one can find that $\langle \Pi_x \rangle = 0$ if $k$ is a complex root of the dispersion equation, and

$$\langle \Pi_x \rangle = -c_g \langle \mathcal{E}_r \rangle, \; c_g = -\frac{1}{2\omega}\frac{\partial \kappa}{\partial k} \tag{2.10}$$

in the region I, where $k = \pm k_0$. The quantity $c_g$ is equal to the projection of the group velocity of the wave on the $x$-axis. The quantity $\langle \mathcal{E}_r \rangle$ is the energy of real monochromatic waves, averaged over their period in the $y$ direction

$$\langle \mathcal{E}_r \rangle = \frac{\omega^2}{2\lambda}|A|^2$$

Since the energy of dissipation is absent in the system, the energy flux of the waves, transporting the energy to infinity is equal to the energy flux of the waves transporting the energy from infinity to the irregularity. Therefore, using the definitions (2.7) and (2.9) one can find

$$|A^+|^2 + |A^-|^2 = |B^+|^2 + |B^-|^2. \tag{2.11}$$

If only one wave transports the energy from $-\infty$ to the irregularity, then

$$A^- = 1, \; A^+ = 0. \tag{2.12}$$

After substituting the formulas (2.3) into the equations (2.5) one finds a system of four linear algebraic equations with respect to the six free constants $a^\pm$ and $a_n$. Adding to this system the equations (2.6) and (2.12), we have an inhomogeneous system of six linear algebraic equations for the six free constants $a^\pm$ and $a_n$.

The transmission and reflection coefficients are equal to $T = B^+$ and $R = B^-$ respectively. From the law of total energy balance (2.11), it follows that

$$|T|^2 + |R|^2 = 1, \; TR^* + RT^* = 0. \tag{2.13}$$

The dependence of the modulus of the transmission coefficient $|T|$ on wave frequency $\omega$ and incident angle $\alpha$, defined by the formula $\sin\alpha = k_y/\lambda_0$, is performed in fig. 3. It is assumed that the Poisson's ratio $\nu = 0.34$. This value is the typical value of the Poisson's ratio of the ice in natural conditions. One can see that the dependence of $|T|$ on $\alpha$ is non monotone for a given value of the wave frequency $\omega$. The quantity $|T|$ is closed

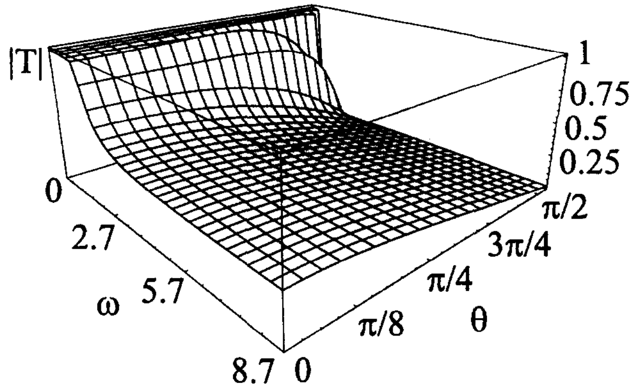


FIGURE 4. The dependence of the modulus of the transmission coefficient on the wave frequency $\omega$ and the angle $\theta$ of wave incidence on the crack. The coefficient $\nu$ is equal to 1.

to unity when $\alpha = \alpha_* \approx 0.45\pi$. The quantity $|T|$ does not depend practically on $\omega$ for $\omega > 3$. If $\alpha \to \pi/2$, then $k_0 \to 0$, $T \to 0$ and $R \to -1$. One can see that the sum of the incident and reflected waves tends to zero in this case. Therefore the full solution of the problem tends to zero as $\alpha \to \pi/2$ also.

In fig. 4 the dependence of $|T|$ on the wave frequency $\omega$ and incident angle $\alpha$ is shown for $\nu = 1$. One can see that the dependence of $|T|$ on $\alpha$ is monotone for a given value of the wave frequency $\omega$. Therefore the process of wave diffraction at the crack strongly depends on the value of the coefficient $\nu$ in the contact-boundary conditions.

Let us consider the properties of the solution (2.4), describing the flexural-gravity oscillations in region II. After substituting (2.4) into (2.5) we find a homogeneous system of four linear algebraic equations with respect to the four free constants $a_n$. One can show that the determinant $\Delta(\omega, k_y)$ of the system is real in region II. For the existence of non zero solutions of this system, its determinant has to be equal to zero:

$$\Delta(\omega, k_y) = 0. \tag{2.14}$$

Any pair of real quantities $\omega$ and $k_y$, satisfying the equation (2.14), defines the frequency and the wave number of the edge wave propagating along the crack. The edge wave transports the energy along the crack, its amplitude decreases exponentially away from the crack. The curve in the plane $(\omega, k_y)$ defined by the equation (2.14) is the dispersion curve of edge waves.

The numerical calculations show that only one branch of the dispersion curve of edge waves exists in the plane $(\omega, k_y)$. This branch, which is related to symmetric edge waves, begins at the origin and lies in a small vicinity of the curve $\omega = \omega_*(k_y)$. The qualitative shape of this branch is shown in fig. 1 by the dotted line.

If the condition (2.14) holds, then all constants $a_n$ can be expressed through one constant $A^e$, which defines the amplitude of edge wave at $x = 0$. The qualitative shape of the fluid surface, deformed by the edge wave with wave number $k_y = k_e$, is shown in fig. 5 in the vicinity of the crack. The function $\zeta(x)$ is shown in fig.6 for the values $\omega = 0.6$(a) and $\omega = 1$(b). The amplitude of the edge wave is normalized to unity. One can see that there are two typical length scales $l_1$ and $l_2$, characterizing the shape of the edge wave profile in the $x$ direction. The length $l_1$, characterizing the scale of damping, is much greater than the wave length $2\pi/k_y$ in the $y$ direction. The scale $l_2 \approx 2\pi/k_e$ defines the distance from the crack edge to the point where the value of $|d^2\zeta/dx^2|$ is maximum.

Thus, it is shown that two modes of flexural-gravity waves exist in deep water beneath an elastic plate with a crack. The asymptotic behavior of the solutions of the first mode away from the crack is associated with plane flexural-gravity waves, transporting the energy from infinity to the crack and from the crack to infinity. The second mode corresponds to symmetric edge waves, propagating along the crack and damping exponentially as they move away from the crack. The waves of various modes do not interact in the linear approximation. Below

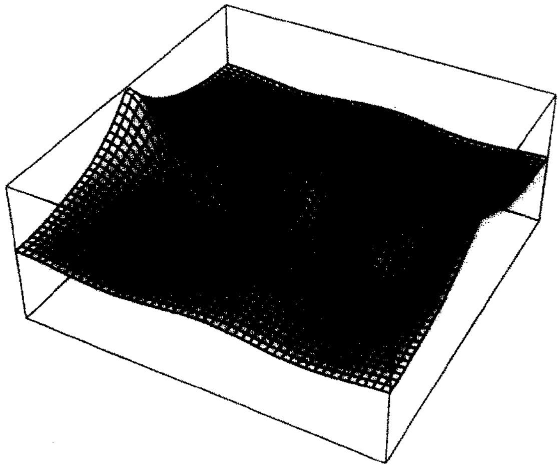

FIGURE 5. The qualitative form of the fluid surface, deformed by the edge wave in the vicinity of the crack.

we will investigate the weakly nonlinear interactions of various wave modes in the system, which lead to energy exchange between them.

## 3. Parametric excitation of edge waves

Assume that the velocity potential $\varphi$ and the perturbation of the fluid surface $\eta$ depend not only on the fast space variables $x$, $y$, $z$ and time $t$, but also on the slow time $\tau = \varepsilon t$. Let us write the following asymptotic expansions

$$\varphi = \sum_{n=0}^{\infty} \varepsilon^n \varphi_n, \quad \eta = \sum_{n=0}^{\infty} \varepsilon^n \eta_n \tag{3.1}$$

where each function $\varphi_n$ satisfies the Laplace equation, the condition $\varphi_n \to 0$ at $z \to -\infty$ and some conditions at $z = 0$.

Let us also consider the following expansions to transform the boundary conditions (1.2) on the fluid surface $z = \varepsilon\eta$ into conditions on the plane $z = 0$,

$$\left.\frac{\partial \varphi}{\partial z}\right|_{z=\varepsilon\eta} = \sum_{n=0}^{\infty} (\varepsilon\eta)^n \left.\frac{\partial^{n+1}\varphi}{\partial z^{n+1}}\right|_{z=0}, \quad \left.\frac{\partial \varphi}{\partial t}\right|_{z=\varepsilon\eta} = \sum_{n=0}^{\infty} (\varepsilon\eta)^n \left.\frac{\partial^{n+1}\varphi}{\partial t \partial z^{n}}\right|_{z=0}. \tag{3.2}$$

Substitute the formulas (3.1) and (3.2) into the conditions (1.2) and set $D = 1$. At order $\varepsilon$ one finds

$$\frac{\partial^2 \varphi_0}{\partial t^2} + (1 + \Delta^2)\frac{\partial \varphi_0}{\partial z} = 0, \quad \frac{\partial \eta_0}{\partial t} = \frac{\partial \varphi_0}{\partial z}, \quad z = 0, \tag{3.3}$$

$$\frac{\partial^2 \varphi_1}{\partial t^2} + (1 + \Delta^2)\frac{\partial \varphi_1}{\partial z} = F_{\varphi}^{(1)}, \quad \frac{\partial \eta_1}{\partial t} = \frac{\partial \varphi_1}{\partial z} + F_{\eta}^{(1)}, \quad z = 0 \tag{3.4}$$

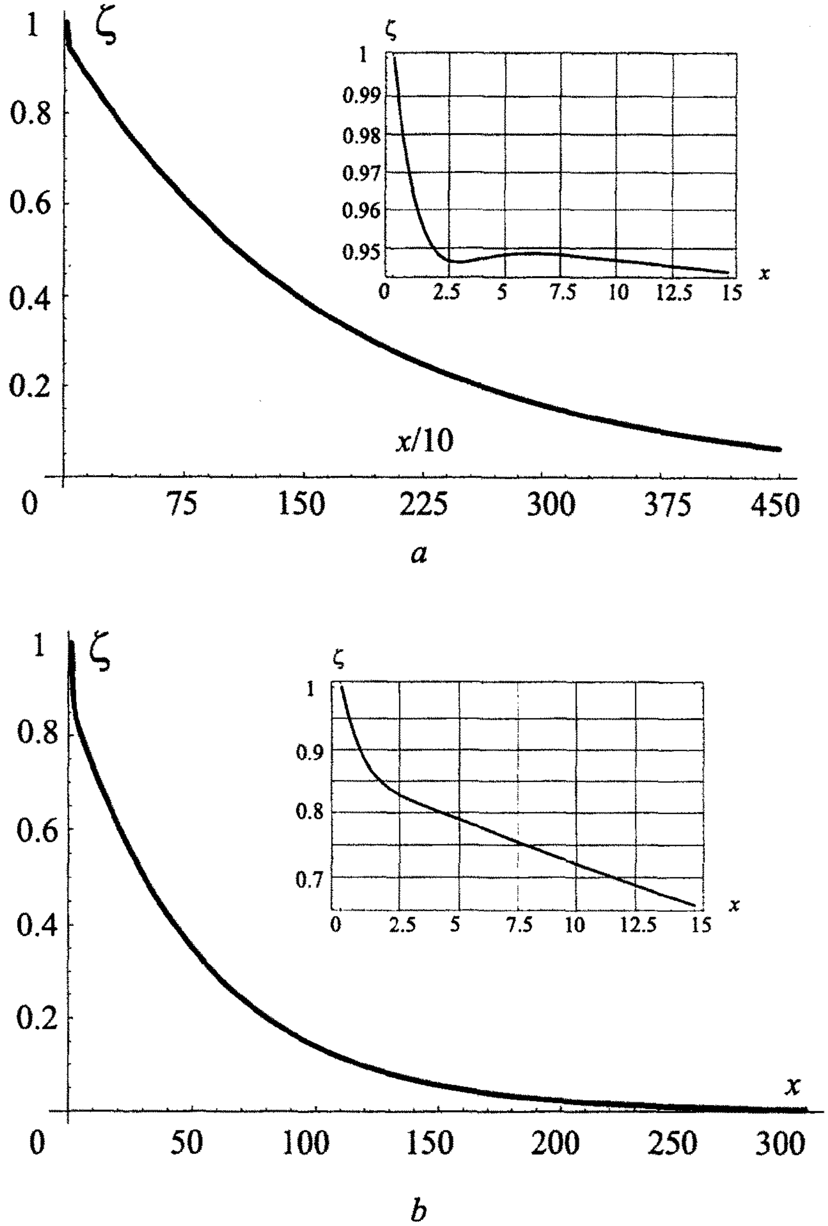


FIGURE 6. The shape of the fluid surface, deformed by the edge wave in the vicinity of the crack. The frequency of the edge wave is: $\omega = 0.6$ (a) and $\omega = 1$ (b).

where the functions $F_{\varphi}^{(1)}$ and $F_{\eta}^{(1)}$ are defined by the formulas

$$
\begin{aligned}
F_{\varphi}^{(1)} &= -2\frac{\partial^2 \varphi_0}{\partial t \partial \tau} - \frac{1}{2}\frac{\partial}{\partial t}\left[2\eta_0 \frac{\partial^2 \varphi_0}{\partial t \partial z} + (\nabla \varphi_0)^2 + \left(\frac{\partial \varphi}{\partial z}\right)^2\right] + (1+\Delta^2)\nabla(\eta_0 \nabla \varphi_0), \\
F_{\eta}^{(1)} &= -\nabla(\eta_0 \nabla \varphi_0) - \frac{\partial \eta_0}{\partial \tau}.
\end{aligned}
\qquad (3.5)
$$

Let us assume that the solution in the linear approximation is the superposition of the incident wave at the crack, the reflected wave, the transmitted wave and two counterpropagating edge waves. This superposition can be represented in the form

$$\begin{aligned} \varphi_0 &= \phi_I e^{i\theta_I} + \phi_{+e} e^{i\theta_{+e}} + \phi_{-e} e^{i\theta_{-e}} + C.C. \\ \eta_0 &= \zeta_I e^{i\theta_I} + \zeta_{+e} e^{i\theta_{+e}} + \zeta_{-e} e^{i\theta_{-e}} + C.C. \end{aligned} \tag{3.6}$$

where $\theta_I = 2\omega^{(0)}t$ and $\theta_{\pm e} = \omega^{(0)}t + k_y^{(0)}y$. The functions $\phi_I$ and $\zeta_I$ are defined by the formulas (2.3), where we set $\omega = 2\omega^{(0)}$, $k_y = 0$ and $P_3(k) = \sum_{n=0}^{4} a_n^I k^n$. The functions $\phi_{\pm e}$ and $\zeta_{\pm e}$ are defined by the formulas (2.4), where $\omega = \omega^{(0)}$, $k_y = \pm k_y^{(0)}$ and $P_3(k) = \sum_{n=0}^{4} a_n^{\pm e} k^n$.

It is assumed that the quantities $a_n^{\pm e}$ are functions of the slow time $\tau$. Moreover $da_n^{\pm e}/d\tau = O(\varepsilon)$, while $da^{\pm}/d\tau = O(\varepsilon^2)$ and $da_n^I/d\tau = O(\varepsilon^2)$. Therefore the dependence of the coefficients $a^{\pm}$ and $a_n^I$ on $\tau$ is not taken into account at order $\varepsilon$.

After substituting the expansions (3.6) into (3.5), we find that the functions $F_\varphi^{(1)}$ and $F_\varphi^{(1)}$ are represented by the expansions

$$\begin{aligned} F_\varphi^{(1)} &= F_{\varphi,I} e^{i\theta_I} + F_{\varphi,+e} e^{i\theta_{+e}} + F_{\varphi,-e} e^{i\theta_{-e}} + C.C. + \cdots, \\ F_\eta^{(1)} &= F_{\eta,I} e^{i\theta_I} + F_{\eta,+e} e^{i\theta_{+e}} + F_{\eta,-e} e^{i\theta_{-e}} + C.C. + \cdots, \end{aligned} \tag{3.7}$$

where the terms defining the solution in the linear approximation (see the formulas (3.6)) are written out. The other terms of the expansions are denoted by the dots. The functions $F_{\varphi,\pm e}$ are defined by the formulas

$$\begin{aligned} F_{\varphi,\pm e} &= \sum_{n=0}^{3} \left[ \frac{da_n^{\pm e}}{d\tau} \frac{\omega^2}{\pi} \int_{-\infty}^{\infty} \frac{k^n e^{ikx} dk}{\kappa(\omega^{(0)}, \lambda)} + a^+ (a_n^{\pm e})^* A_+^{\pm n}(x) + a^- (a_n^{\mp e})^* A_-^{\pm n}(x) \right] + \sum_{n,m=0}^{3} a_m^I (a_n^{\mp e})^* A^{\pm mn}(x), \\ F_{\varphi,I}(x) &= \sum_{n,m=0}^{3} a_n^{+e} a_m^{-e} A_I^{mn}(x). \end{aligned} \tag{3.8}$$

The coefficients $A_+^{\pm n}$, $A_-^{\pm n}$, $A^{\pm mn}(x)$ and $A_I^{mn}(x)$ can be simply calculated by substituting the expressions (3.6) - (3.8) into the formula (3.5). Each of them is bounded, damps exponentially at $|x| \to \infty$ and can have discontinuities as the point $x = 0$.

The functions $F_{\eta,\pm e}$ and $F_{\eta,I}$ depend on $a^{\pm}$, $a_n^I$, $a_n^{\pm e}$ and $da_n^{\pm e}/dt$ in the same way as the functions $F_{\varphi,\pm e}$ and $F_{\varphi,I}$. Their Fourier images do not have singularities on the real axis of the spectral parameter $k$. From this, it follows that the dependence of the functions $\varphi_1$ and $\eta_1$ on $a^{\pm}$, $a_n^I$, $a_n^{\pm e}$ and $da_n^{\pm e}/dt$ is represented by formulas similar to (3.8) with accuracy $O(\varepsilon)$. In particular, one can write

$$\eta_1 = \zeta_{1,I} e^{i\theta_I} + \zeta_{1,+e} e^{i\theta_{+e}} + \zeta_{1,-e} e^{i\theta_{-e}} + C.C. + \cdots \tag{3.9}$$

where

$$\begin{aligned} \zeta_{1,I} &= \sum_{n,m=0}^{3} a_n^{+e} a_m^{-e} B_I^{nm}(x), \\ \zeta_{1,\pm e} &= \sum_{n=0}^{3} \left[ \frac{da_n^{\pm e}}{d\tau} B^n(x) + a^+ (a^{\mp e})^* B_+^{\pm n}(x) + a^- (a^{\mp e})^* B_-^{\pm n}(x) \right] + \sum_{n,m=0}^{3} a_m^I (a_n^{\mp e})^* B^{\pm mn}(x). \end{aligned} \tag{3.10}$$

Each of the coefficients $B^n(x)$, $B_+^{\pm n}(x)$, $B_-^{\pm n}(x)$, $B^{\pm mn}(x)$ and $B_I^{mn}(x)$ is bounded, damps exponentially as $|x| \to \infty$ and can have discontinuities at the point $x = 0$. The dots in the formula (3.9) mean that in the expansion for the function $\eta_1$ only the terms defining the solution in the linear approximation (see the formulas (3.6)) are written out.

From the formulas (3.1), (3.6), (3.9) and (3.10) follows the expression for the perturbation of fluid surface. With accuracy $O(\varepsilon)$ we have

$$\eta = (\zeta_I + \varepsilon\zeta_{1,I})e^{i\theta_I} + (\zeta_{+e} + \varepsilon\zeta_{1,+e})e^{i\theta_{+e}} + (\zeta_{-e} + \varepsilon\zeta_{1,-e})e^{i\theta_{-e}} + C.C. + \cdots \tag{3.11}$$

where the meaning of the dots is the same as in the formula (3.9).

Let us substitute the expression (3.11) into the contact-boundary conditions (1.7), multiply the resulting equations by $e^{-i\theta_I}$ and then average them over the fast time $t$. Repeat the averaging, by multiplying the contact-boundary conditions by $e^{-i\theta_{+e}}$ and $e^{-i\theta_{-e}}$. We have as a result

$$\begin{gathered}\lim_{x\to\pm0}\frac{\partial^2}{\partial x^2}(\zeta_I + \varepsilon\zeta_{1,I}) = \lim_{x\to\pm0}\frac{\partial^3}{\partial x^3}(\zeta_I + \varepsilon\zeta_{1,I}) = 0, \\ \lim_{x\to\pm0}\left[\frac{\partial^2}{\partial x^2} - \nu(k_y^{(0)})^2\right](\zeta_{\pm e} + \varepsilon\zeta_{1,\pm e}) = \lim_{x\to\pm0}\frac{\partial}{\partial x}\left[\frac{\partial^2}{\partial x^2} - \nu'(k_y^{(0)})^2\right](\zeta_{\pm e} + \varepsilon\zeta_{1,\pm e}) = 0.\end{gathered} \tag{3.12}$$

Taking into account the formulas (2.3), (2.4) and (3.10) one can find that the twelve equations (3.12) have the following form

$$a^+C_j^+ + a^-C_j^- + \sum_{n=0}^{3} a_n^I C_{j,n}^I + \varepsilon \sum_{m,n=0}^{3} a_m^{+e} a_n^{-e} C_{j,mn}^I = 0, \tag{3.13}$$

$$\sum_{n=0}^{3} a_n^{\pm e} C_{j,n} + \varepsilon \sum_{n=0}^{3}\left[\frac{da_n^{\pm e}}{d\tau} D_{j,n} + a^+(a_n^{\mp e})^* D_{j,n}^+ + a^-(a_n^{\mp e})^* D_{j,n}^-\right] + \varepsilon \sum_{m,n=0}^{3} a_m^I (a_n^{\pm e})^* D_{j,mn}^{\pm e} = 0,\ j = 1,\cdots,4 \tag{3.14}$$

where all the coefficients, denoted by the letters $C$ and $D$ with subscripts and superscripts, are some real numbers.

The system of twelve equations (3.13) and (3.14) consists of fourteen unknown functions $a^\pm$, $a_n^I$ and $a_n^{\pm e}$ ($n = 0 - 3$) and must be completed by the equations (2.6), which define the amplitudes $A^\pm$ of the waves, transporting the energy from infinity to the crack.

In the linear approximation the equations (2.6) and (3.13) form the linear system of algebraic equations, the solution of which defines the linear dependence of the constants $a^\pm$ and $a_n^I$ on the wave amplitudes $A^\pm$. In the special case, where only one wave transports the energy from $-\infty$ to the crack, the constants $a^\pm$ and $a_n^I$ are proportional to the amplitude $A^-$

$$a^\pm = c^\pm A^-,\ a_n^I = c_n^I A^- \tag{3.15}$$

In the linear approximation the equations (3.14) form two homogeneous systems of linear algebraic equations with determinants equal to zero (see the equation (2.14)). The solution of these systems can be written in the form

$$a_n^{\pm e} = c_n A^{\pm e} \tag{3.16}$$

where $A^{\pm e}$ are the amplitudes of the edge waves and $c_n$ are the proportionality factors.

After substituting the expressions (3.15) and (3.16) into (3.14) one can find the equations of the parametric excitation of counterpropagating edge waves by normally incident flexural-gravity waves on the crack

$$\frac{dA^{\pm e}}{d\tau} = \alpha A^-(A^{\mp e})^*,\ \alpha = \text{const}. \tag{3.17}$$

The solution of (3.17) has the following form

$$A^{\pm e} = A_0^{\pm e} e^{\gamma\tau}, \ \gamma = |\alpha A^-|^2.$$

One can see that the increment of the growth of edge wave amplitudes is proportional to the energy of the wave normally incident on the crack. The extension of the edge waves will change the reflection coefficient of the wave incident on the crack. This change will be of order one, when the amplitude of the edge wave will be of order $1/\sqrt{\varepsilon}$. At that time the assumption about the independence of the coefficients $a^{\pm}$ and $a_n^I$ on the slow time $\tau$ loses its meaning and the theory is not longer applicable.

## Conclusions

It is shown that the transmission coefficients of the flexural-gravity waves through the crack strongly depend on the coefficient $\nu$ in the contact-boundary conditions at the crack edges. In the special case where the classic model of thin elastic plate constructed on the basis of the Kirchhoff-Love hypotheses and Hook's law is used, the dependence of the modulus of the wave transition coefficient on the incident angle of the wave $\alpha$ at the crack is non monotone for a given value of wave frequency. The wave reflection from the crack is essentially absent, when $\alpha = \alpha_* \approx 0.45\pi$ . The use of other models of the floating plate can lead to a monotonic form of this dependence.

In natural conditions the existence of this effect can induce a preferred filtration of flexural-gravity waves in an anisotropic ice cover with a given orientation of the cracks. In this case the coefficient $\nu$ in the contact-boundary conditions, which is the large scale analogue of the Poisson's ratio, can be defined by the angle between the crack direction and the direction of wave numbers, related to slowly damped flexural-gravity waves.

The existence of the angle $\alpha_*$ is the consequence of the form of contact-boundary conditions at the crack edges only. The value of the angle $\alpha_*$ does not depend on the value of the ice rigidity $D$. In this sense this effect is different from the effect of the existence of critical angle of wave incidence at the edge of a half infinite floating elastic plate (Squire et al., 1995), the existence of which depend on the properties of the dispersion equations for gravity waves and flexural-gravity waves. Recall that if the angle of wave incidence at the ice edge is larger than the critical value, then the wave transition coefficient is equal to zero.

In the case of wave diffraction at the crack the transition coefficient tends to zero, when the incident angle of the wave tends to $\pi/2$. Therefore the cracks, which are parallel to the wave vectors of flexural-gravity waves, strongly scatter the wave energies. Apparently, the existence of space nonhomogeneity directed along the cracks can induce the transmission of the wave energy into the energy of edge waves, running along the cracks.

The curvature of the surface of elastic plate, deformed by the edge wave in the vicinity of the crack, has sharp maxima on a distance approximately equal to the wave length. This effect can reduce the fracture of ice edges in the vicinity of the crack. Note that the formulas (2.4), describing symmetric edge waves, can also describe edge waves propagating along a vertical wall. The vertical wall can model the side of a ship or shelf construction. The edge waves excitation near the shelf construction can induce the fracture of the ice in the vicinity of the wall and stipulate moreover the lowering of ice pressure on the wall.

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